Partial hyperbolicity and ergodicity on $\mathbb{T}^3$

RAÚL URES

Universidad de la República, Uruguay

Luminy, 2011

Definitions

- $f : M \to M$ is partially hyperbolic if

$$TM = E^s \bigoplus E^c \bigoplus E^u$$

such that E^σ_x are invariant and $v^\sigma \in E^\sigma_x$ (unit vectors) satisfy: ($\sigma = s, c, u$)

Definitions

- $f : M \to M$ is partially hyperbolic if

$$TM = E^s \oplus E^c \oplus E^u$$

  such that E^σ_x are invariant and $v^\sigma \in E^\sigma_x$ (unit vectors) satisfy: ($\sigma = s, c, u$)

- $\|Df(x)v^s\| < \|Df(x)v^c\| < \|Df(x)v^u\|$

Definitions

- $f : M \to M$ is partially hyperbolic if

$$TM = E^s \oplus E^c \oplus E^u$$

such that E_x^σ are invariant and $v^\sigma \in E_x^\sigma$ (unit vectors) satisfy: $(\sigma = s, c, u)$

- $\|Df(x)v^s\| < \|Df(x)v^c\| < \|Df(x)v^u\|$
- $\|Df(x)v^s\| < 1 < \|Df(x)v^u\|$

Definitions

- $f : M \to M$ is partially hyperbolic if

$$TM = E^s \bigoplus E^c \bigoplus E^u$$

such that E^σ_x are invariant and $v^\sigma \in E^\sigma_x$ (unit vectors) satisfy: $(\sigma = s, c, u)$

- $\|Df(x)v^s\| < \|Df(x)v^c\| < \|Df(x)v^u\|$
- $\|Df(x)v^s\| < 1 < \|Df(x)v^u\|$
- In this talk, f will be always conservative (volume preserving)

Accessibility

- It is well-known that the stable and unstable sub-bundles are uniquely integrable.

Accessibility

- It is well-known that the stable and unstable sub-bundles are uniquely integrable.
-

DEFINITION . *The accessibility class of a point x is the set of points that can be joined with x by a path that is tangent to $E^s \cup E^u$.*

Accessibility

- Ergodicity is strongly related with accessibility

Accessibility

- Ergodicity is strongly related with accessibility
- f is accessible (or has the accessibility property) if the whole manifold is the unique accessibility class

Accessibility

- Ergodicity is strongly related with accessibility
- f is accessible (or has the accessibility property) if the whole manifold is the unique accessibility class
- f is essentially accessible if any measurable set that is a union of accessibility classes has null or full volume

Previous results and motivations

In this talk our manifold will be 3-dimensional ($\mathbb{T}^3$)

Previous results and motivations

In this talk our manifold will be 3-dimensional ($\mathbb{T}^3$)

THEOREM (BW, HHU). *.*

(Essential) Accessibility $\Rightarrow$ *ergodicity*

Previous results and motivations

In this talk our manifold will be 3-dimensional ($\mathbb{T}^3$)

THEOREM (BW, HHU). *.*

(Essential) Accessibility $\Rightarrow$ *ergodicity*

CONJECTURE (PS). *Ergodicity is open and dense among the conservative partially hyperbolic diffeomorphisms.*

Previous results and motivations

THEOREM (HHU). *Pugh-Shub Conjecture is true if* $dim(E^c) = 1$*, in particular, if* $dim(M) = 3$

Previous results and motivations

THEOREM (HHU). *Pugh-Shub Conjecture is true if* $dim(E^c) = 1$, *in particular, if* $dim(M) = 3$

QUESTION . *Is it possible to give a more accurate description of the ergodic PH diffeomorphisms in dimension 3?*

Previous results and motivation

CONJECTURE (HHU). *If M supports a non-ergodic PH diffeomorphism then it is one of the following manifolds:*

1. $\mathbb{T}^3$
2. *Mapping torus of* $-Id$
3. *Mapping torus of a hyperbolic automorphism of* $\mathbb{T}^2$

Previous results and motivation

CONJECTURE (Stronger version). *If f is a non ergodic PH diffeomorphism then, there is a torus tangent to $E^s \bigoplus E^u$.*

Previous results and motivation

CONJECTURE (Stronger version). *If f is a non ergodic PH diffeomorphism then, there is a torus tangent to $E^s \bigoplus E^u$.*

Jana has already explained that the stronger version implies the conjecture (HHU)

Previous results and motivation

CONJECTURE (Stronger version). *If* f *is a non ergodic PH diffeomorphism then, there is a torus tangent to* $E^s \bigoplus E^u$.

Jana has already explained that the stronger version implies the conjecture (HHU)

In this talk we will show some advances in the direction of the stronger version of the conjecture when $M = \mathbb{T}^3$

Previous results and motivations

In $\mathbb{T}^3$ it is possible to construct non-ergodic examples if f is not homotopic to Anosov.

Previous results and motivations

In $\mathbb{T}^3$ it is possible to construct non-ergodic examples if f is not homotopic to Anosov.

Indeed, the corresponding linear automorphism itself is not ergodic.

Previous results and motivations

In $\mathbb{T}^3$ it is possible to construct non-ergodic examples if f is not homotopic to Anosov.

Indeed, the corresponding linear automorphism itself is not ergodic.

Then, it is natural to ask what happens when f is homotopic to a linear hyperbolic automorphism.
From now on, we will suppose that f is homotopic to Anosov

Main result

THEOREM (Hammerlindl, U.). .

1. *f is not accessible $\Rightarrow$ f is conjugated to Anosov.*

Main result

THEOREM (Hammerlindl, U.). .

1. *f is not accessible $\Rightarrow$ f is conjugated to Anosov.*

2. *f is not ergodic $\Rightarrow$ $\lambda_c(x) = 0$ for a.e. x*

Sketch of the proof

The proof of the first part of the Theorem is divided in three steps.

Sketch of the proof

The proof of the first part of the Theorem is divided in three steps.

1. $E^s \oplus E^u$ integrates to a minimal foliation

Sketch of the proof

The proof of the first part of the Theorem is divided in three steps.

1. $E^s \oplus E^u$ integrates to a minimal foliation
2. This foliation admits a (unique up to multiplying by a constant) transverse holonomy invariant measure μ with full support

Sketch of the proof

The proof of the first part of the Theorem is divided in three steps.

1. $E^s \oplus E^u$ integrates to a minimal foliation
2. This foliation admits a (unique up to multiplying by a constant) transverse holonomy invariant measure μ with full support
3. $f_* \mu = \lambda \mu$ with $\lambda \neq 1$

Sketch of the proof

The proof of the first part of the Theorem is divided in three steps.

1. $E^s \oplus E^u$ integrates to a minimal foliation
2. This foliation admits a (unique up to multiplying by a constant) transverse holonomy invariant measure μ with full support
3. $f_*\mu = \lambda\mu$ with $\lambda \neq 1$

This last step will imply that f has local product structure and Franks' arguments give the conjugacy with the linear part.

First step

PROPOSITION (HHU). *If f is not accessible we have three possibilities:*

First step

PROPOSITION (HHU). *If f is not accessible we have three possibilities:*

1. *there is a 2-torus tangent to $E^s \oplus E^u$*

First step

PROPOSITION (HHU). *If f is not accessible we have three possibilities:*

1. *there is a 2-torus tangent to $E^s \oplus E^u$*
2. *there is a proper lamination tangent to $E^s \oplus E^u$ that extends to a foliation without compact leaves*

First step

PROPOSITION (HHU). *If* f *is not accessible we have three possibilities:*

1. *there is a 2-torus tangent to* $E^s \oplus E^u$
2. *there is a proper lamination tangent to* $E^s \oplus E^u$ *that extends to a foliation without compact leaves*
3. $E^s \oplus E^u$ *integrates to a minimal foliation*

First step

- It is not possible to have invariant 2-torus because f is homotopic to Anosov

First step

- It is not possible to have invariant 2-torus because f is homotopic to Anosov
- The case of the proper lamination is not possible for f on $\mathbb{T}^3$.

First step

- It is not possible to have invariant 2-torus because f is homotopic to Anosov
- The case of the proper lamination is not possible for f on $\mathbb{T}^3$.
- Indeed, boundary leaves of the lamination are periodic and have an Anosov-like dynamics with dense periodic points.

First step

- Since the lamination extends to a foliation without compact leaves, the inclusion of any leaf F injects its fundamental group in $\pi_1(\mathbb{T}^3)$. Then, the fundamental group of each leaf is abelian.

First step

- Since the lamination extends to a foliation without compact leaves, the inclusion of any leaf F injects its fundamental group in $\pi_1(\mathbb{T}^3)$. Then, the fundamental group of each leaf is abelian.
- Observe that the unique surface with abelian fundamental group that supports such an Anosov dynamics is the 2-torus

First step

- Since the lamination extends to a foliation without compact leaves, the inclusion of any leaf F injects its fundamental group in $\pi_1(\mathbb{T}^3)$. Then, the fundamental group of each leaf is abelian.

- Observe that the unique surface with abelian fundamental group that supports such an Anosov dynamics is the 2-torus

- Then, the unique possibility is that $E^s \oplus E^u$ integrates to a minimal foliation

Second step

- Plante has proved that if the fundamental group of a 3-dimensional manifold has sub-exponential growth (transversally orientable) foliations have holonomy invariant measures. If the foliation has no compact leaves then, each minimal set of the foliation has one such a measure (up to a constant factor)

Second step

- Plante has proved that if the fundamental group of a 3-dimensional manifold has sub-exponential growth (transversally orientable) foliations have holonomy invariant measures. If the foliation has no compact leaves then, each minimal set of the foliation has one such a measure (up to a constant factor)
- In our case, since the foliation is minimal, we have a unique holonomy invariant measure μ and it has full support

Second step

- Plante has proved that if the fundamental group of a 3-dimensional manifold has sub-exponential growth (transversally orientable) foliations have holonomy invariant measures. If the foliation has no compact leaves then, each minimal set of the foliation has one such a measure (up to a constant factor)
- In our case, since the foliation is minimal, we have a unique holonomy invariant measure μ and it has full support
- Moreover, Plante also proved that these measures represent nontrivial elements of the first cohomology group.

Third step

- Since the foliation is f-invariant, the push forward of μ is a holonomy invariant measure.

Third step

- Since the foliation is f-invariant, the push forward of μ is a holonomy invariant measure.
- Then, it is a multiple of μ by a factor λ.

Third step

- Since the foliation is f-invariant, the push forward of μ is a holonomy invariant measure.
- Then, it is a multiple of μ by a factor λ.
- In other words μ represents an eigenvector of the action of f in the cohomology.

Third step

- Since the foliation is f-invariant, the push forward of μ is a holonomy invariant measure.
- Then, it is a multiple of μ by a factor λ.
- In other words μ represents an eigenvector of the action of f in the cohomology.
- This action is hyperbolic and then, $\lambda \neq 1$

End of the proof

- Suppose that $\lambda < 1$.

End of the proof

- Suppose that $\lambda < 1$.
- Then, if γ is a center curve (that is always transverse to $E^s \oplus E^u$) $\mu(f^n(\gamma)) \to 0$.

End of the proof

- Suppose that $\lambda < 1$.
- Then, if γ is a center curve (that is always transverse to $E^s \oplus E^u$) $\mu(f^n(\gamma)) \to 0$.
- The angle between E^c and $E^s \oplus E^u$ is uniformly bounded then, the length of γ goes to 0.

End of the proof

- Suppose that $\lambda < 1$.
- Then, if γ is a center curve (that is always transverse to $E^s \oplus E^u$) $\mu(f^n(\gamma)) \to 0$.
- The angle between E^c and $E^s \oplus E^u$ is uniformly bounded then, the length of γ goes to 0.
- This implies that $E^c \oplus E^s$ is uniquely integrable and that f has local product structure.

End

THANK YOU