Symbolic extensions on surfaces in intermediate smoothness

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E.N.S. Cachan, France

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Symbolic extensions VS Smoothness

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$T : M \to M$ a $\mathcal{C}^r$ map with a real $r \geq 1$.

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Goal

Prove the existence of symbolic extensions and more precisely estimate the symbolic extension entropy, $h_{sex}(T) := \inf_S h_{top}(S)$, *in terms of*

- *the topological entropy of* T*;*
- *the smoothness, i.e. the parameter* r*;*
- *the growth of the first derivative.*

Overview of known results and the Dow-New conjecture

- $\mathcal{C}^\infty$ dynamical systems always admit principal symbolic extensions, i.e. there exist symbolic extensions $\pi : (Y, S) \to (X, T)$ s.t. $\forall \nu \in \mathcal{M}(Y, S),\ h(\pi_* \nu) = h(\nu)$ (Boy-Fie-Fie);

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Conjecture (Dow-New)

Any $\mathcal{C}^r$ map $T : M \to M$ with $r > 1$ admits a symbolic extension. Moreover

$$h_{sex}(T) \leq h_{top}(T) + \frac{dR(T)}{r-1}$$

with $R(T) = \lim_n \frac{1}{n} \log^+ \|DT^n\|_\infty$.

SEX theorem

- Extension entropy of a symbolic extension $\pi : (Y, S) \to (X, T)$

$$\begin{array}{rcl} h^{\pi}_{ext} : \mathcal{M}(X,T) & \to & \mathbb{R}^+ \\ \mu & \mapsto & \sup_{\pi_*\nu=\mu} h(\nu) \end{array}$$

- Usc envelope of an upper bounded function

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Theorem (Boy-Dow)

Let $(h_k)_k$ be an entropy structure of (X, T). The functions $h_{ext}^{\pi} - h$ are the affine usc functions g satisfying $\widetilde{\lim_k g + h - h_k} = g$.

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Observe that if (X, T) has a symbolic extension, then h is a difference of nonnegative affine usc functions.

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Conjecture

For any $\mathcal{C}^r$ map with $r > 1$, h is a difference of nonnegative usc functions.

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Question

Do there exist smooth dynamical systems with h usc and $h_{sex}(T) > h_{top}(T)$?

Find an affine usc function g satisfying $\lim_k \widetilde{g + h - h_k} = g$.

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Lemma

$\overline{\sum_i \chi_i^+} : \mu \mapsto \int \sum_i \chi_i^+(x) d\mu(x)$ *is an affine upper semicontinuous function on* $\mathcal{M}(M, T)$.

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<u>Proof :</u>

$$\begin{aligned}
\overline{\sum_i \chi_i^+}(\mu) &= \int \sum_i \chi_i^+(x) d\mu(x) \\
&= \int \lim_n \frac{1}{n} \max_k \log^+ \|\Lambda^k D_x T^n\|_k d\mu(x) \text{ (Oseledets)} \\
&= \lim_n \frac{1}{n} \int \max_k \log^+ \|\Lambda^k D_x T^n\|_k d\mu(x) \\
&= \inf_n \frac{1}{n} \int \max_k \log^+ \|\Lambda^k D_x T^n\|_k d\mu(x) \text{ (subadditivity)}
\end{aligned}$$

Statements

Question

Does the function $g = \overline{\frac{\sum_i \chi_i^+}{r-1}}$ *satisfies* $\lim_k g \widetilde{+ h -} h_k = g$ *?*

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Main Proposition (B.)

$\forall \mu \in \mathcal{M}(M, T)\ \forall \gamma > 0\ \exists \delta_\mu > 0\ \exists k_\mu \in \mathbb{N}$
$\forall \nu \in \mathcal{M}\ (M, T)$ *with* $dist(\nu, \mu) < \delta_\mu$

$$(h - h_{k_\mu})(\nu) \leq \frac{}{r-1}\left(\overline{\sum_i \ \chi_i^+}(\mu) - \sum_i \chi_i^+(\nu)\right) + \gamma$$

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Corollary (Dow-Maa)

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For any $\mathcal{C}^r$ surface diffeomorphim (resp. noninvertible map) with $r > 1$ there exists a symbolic extension with $h^{\pi}_{ext} - h = \frac{\overline{\chi_1^+}}{r-1}$ (resp. $\frac{2\left(\overline{\sum_i \chi_i^+}\right)}{r-1}$).

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The estimates of the extension entropy of $\mathcal{C}^r$ interval maps and $\mathcal{C}^r$ surface diffeomorphisms are sharp according to the pathological examples of B. and Dow-New. More precisely, for these dynamical systems (M, T), there exists an invariant measure μ such that $(h^{\pi}_{ext} - h)(\mu) \geq \frac{1}{r-1} R(T)$ for all symbolic extensions $\pi : (Y, S) \to (X, T)$.

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Let's go sketch the proof of the main Proposition for... $l_\nu = 1$.

Newhouse local entropy and local volume growth

$\sigma : [0,1]^k \to M$ of class $\mathcal{C}^r$ with $1 \leq k \leq d$, $\epsilon > 0$ and $F \subset M$,

$$v(\sigma, \epsilon, F) := \limsup_n \frac{1}{n} \ln^+ \sup_{x \in F} \int_{\sigma^{-1} B(x,n,\epsilon)} \|\Lambda^k D_y(T^n \circ \sigma)\|_k \, d\lambda(y)$$

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Definition

$\nu \in \mathcal{M}_e(M,T)$,

$$v(\nu,\epsilon) := \lim_{\alpha\to 1} \inf_{\nu(F_\alpha)>\alpha} \sup_{\substack{\sigma:[0,1]^{l_\nu}\to M \\ \max_{k=1,\dots,r} \|D^k\sigma\|_\infty \leq 1}} v(\sigma,\epsilon,F_\alpha)$$

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Theorem (New)

$\forall \epsilon > 0$, $\forall \nu \in \mathcal{M}_e(M, T)$,

$$h^{New}(\nu, \epsilon) \leq v(\nu, \epsilon)$$

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Proposition

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To prove the Main Proposition for $l_\nu = 1$ it is enough to prove the following one

Proposition

$\forall \mu \in \mathcal{M}(M, T)$ $\exists \delta_\mu > 0$ $\exists \epsilon_\mu \in \mathbb{N}$
$\forall \nu \in \mathcal{M}_e(M, T)$ *with* $dist(\nu, \mu) < \delta_\mu$ *and* $l_\nu = 1$

$$v(\nu, \epsilon_\mu) \lesssim \frac{1}{r-1}\left(\overline{\chi_1^+}(\mu) - \chi_1^+(\nu)\right)$$

Reparametrization Lemma

Lemma

Let $T : M \to M$ *be a* $\mathcal{C}^r$ *map with* $r > 1$, *then* $\exists \epsilon > 0$ *s.t.*
$\forall \sigma : [0,1] \to M$ *with* $\max_{1 \leq k \leq r} \|D^k \sigma\| \leq 1$
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- $\frac{\log \sharp \mathcal{F}_n}{n} \lesssim \frac{1}{r-1}\left(\frac{1}{n}\sum_{k=0}^{n-1} \log^+ \|D_{T^k x} T\| - \chi\right)$.

Proof of the Main Proposition for $l_\nu = 1$ assuming the Reparametrization Lemma

Let $\mu \in \mathcal{M}(M,T)$, we want to prove there exist $\delta_\mu > 0$ and $\epsilon_\mu > 0$ s.t. for ergodic measure ν δ_μ-close to μ

$$v(\nu, \epsilon_\mu) \lesssim \frac{1}{r-1}\left(\overline{\chi_1^+}(\mu) - \chi_1^+(\nu)\right)$$

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$$\overline{\chi_1^+}(\mu) = \inf_n \frac{1}{n} \int \log^+ \|D_x T^n\| d\mu(x)$$

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Choose $\delta_\mu > 0$ s.t. for ν δ_μ-close to μ so that

$$\int \log^+ \|D_x T\| d\nu(x) \simeq \int \log^+ \|D_x T\| d\mu(x)$$

- Choice of ϵ_μ : Apply the Reparametrization Lemma to T, you get ϵ_μ such that for all subset F of M, for all $\sigma : [0,1] \to M$ with $\max_{1\leq k\leq r} \|D^k\sigma\| \leq 1$,

$$v(\sigma, \epsilon_\mu, F, \chi_1^+(\nu)) \lesssim \overline{\lim}_n \frac{1}{n} \sup_{x\in F} \frac{1}{r-1}\left(\frac{1}{n}\sum_{k=0}^{n-1} \log^+ \|D_{T^k x}T\| - \chi_1^+(\nu)\right)$$

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- Choice of F_α : For all $\alpha < 1$ we choose by the Ergodic Theorem a Borel set F_α with $\nu(F_\alpha) > \alpha$ s.t. $\left(\frac{1}{n}\sum_{k=0,\dots,n-1} \log^+ \|D_{T^k x} T\|\right)_n$ converges uniformly in $x \in F$ to $\int \log^+ \|D_x T\| d\nu(x)$.

- Choice of ϵ_μ : Apply the Reparametrization Lemma to T, you get ϵ_μ such that for all subset F of M, for all $\sigma : [0,1] \to M$ with $\max_{1\leq k\leq r} \|D^k\sigma\| \leq 1$,

$$v(\sigma, \epsilon_\mu, F, \chi_1^+(\nu)) \lesssim \overline{\lim}_n \frac{1}{n} \sup_{x\in F} \frac{1}{r-1} \left(\frac{1}{n} \sum_{k=0}^{n-1} \log^+ \|D_{T^k x} T\| - \chi_1^+(\nu) \right)$$

- Choice of F_α : For all $\alpha < 1$ we choose by the Ergodic Theorem a Borel set F_α with $\nu(F_\alpha) > \alpha$ s.t. $\left(\frac{1}{n}\sum_{k=0,\dots,n-1} \log^+ \|D_{T^k x} T\|\right)_n$ converges uniformly in $x \in F$ to $\int \log^+ \|D_x T\| d\nu(x)$.

- Conclusion : We get finally for ergodic measures ν δ_μ-close to μ,

$$\begin{aligned} v(\nu, \epsilon_\mu) &\lesssim \frac{1}{r-1} \left(\int \log^+ \|D_x T\| d\nu(x) - \chi_1^+(\nu) \right) \\ &\lesssim \frac{1}{r-1} \left(\overline{\chi_1^+}(\mu) - \chi_1^+(\nu) \right) \end{aligned}$$

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For a $\mathcal{C}^r$ curve $\sigma : [0,1] \to \mathbb{R}^d$, we want estimate the local length, i.e the length of $\sigma|_{\sigma^{-1}(B(0,1))}$.

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To estimate the local length of a $\mathcal{C}^r$ general curve, we proceed as follows. Cut the interval $[0,1]$ into $[\|D^r\sigma\|^{\frac{1}{r}}] + 1$ subintervals of size less than $\frac{1}{\|D^r\sigma|^{\frac{1}{r}}}$. Reparametrize these intervals by affine contractions ϕ. Then we have $\|D^r\sigma \circ \phi\|_\infty \leq 1$. By applying the above lemma, the local length of σ is bounded from above by $\|D^r\sigma\|^{\frac{1}{r}} + 1$ up to a multiplicative constant.

In fact we want estimate the volume of $\sigma|_{\sigma^{-1}(B(0,1))\cap\{\frac{a}{2}\leq\|\sigma'\|\leq 2a\}}$.

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Lemma

Assume $\forall t,s\in[0,1],\ \|\sigma'(t)-\sigma'(s)\|\leq\frac{\|\sigma'\|_\infty}{3}$ *then* $\exists\mathcal{F}$ *a family of (affine) maps s.t.*

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For a general curve σ we proceed as follows. We first apply the following one parameter version of Gromov-Yomdin Lemma for σ' and $r-1$ to cut the interval $[0,1]$ into subintervals on which $\|\sigma'(t)-\sigma'(s)\|\leq\frac{a}{6}$. Then the volume $\sigma|_{\sigma^{-1}(B(0,1))\cap\{\frac{a}{2}\leq\|\sigma'\|\leq 2a\}}$ is bounded from above by the number of subintervals up to a multiplicative constant.

Lemma (Gromov-Yomdin)

Assume $\|D^r\sigma\|_\infty \leq 1 \leftarrow a$ *then* $\exists \mathcal{F} = (\phi : [0,1] \to [0,1])$ *a family of (s.a.) maps s.t.*

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Assume $\|D^r\sigma\|_\infty \leq 1 \leftarrow a$ *then* $\exists \mathcal{F} = (\phi : [0,1] \rightarrow [0,1])$ *a family of (s.a.) maps s.t.*

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We conclude that the volume of $\sigma|_{\sigma^{-1}(B(0,1))\cap\{\frac{a}{2}\leq\|\sigma'\|\leq 2a\}}$ is bounded from above by

$[\left(\frac{\|D^r\sigma\|_\infty}{a}\right)^{\frac{1}{r-1}}]+1$ up to a multiplicative constant.

The induction

We are working in local small charts along the orbit of x :

$$T_n = \exp_{T^n x}^{-1}(\epsilon^{-1}.) \circ T \circ \exp_{T^{n-1}x}(\epsilon.)$$

One can choose ϵ small enough $\|D^s T_n\|_\infty \leq 1$ for all $2 \leq s \leq r$.

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Painful! It involves in particular the s-derivative of $T^{n-1} \circ \sigma$ of order less than r, but they are related with the r derivative by using Landau-Kolmogorov type inequalities.